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Generalization of Einstein's Equation for Nonsymmetric $T_{\mu\nu}$

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Abstract

We show how our hypercomplex number approach to the basic laws (which works so well for Dirac and Maxwell theory) leads to a divergence from the Einstein approach to gravity except in the contracted scalar equation. Our equation seems to allow a nonsymmetric $T_{\mu\nu}$ source tensor. The serious problem of energy conservation is considered but not resolved.

1. *Introduction*

Using "equivalence" speculations, Einstein surmised that gravitation, as an interaction, is best looked at as a warping of the space-time in which bodies move freely. He conjectured that all forms of energy concentration contribute to the curvature in their vicinity. This was a very bold guess and totally unsupported by any existing data at the time. He was able to predict effects of the curvature near the sun which were subsequently verified. The tests are macroscopic (light bending and perihelion precession) or "qualitative" ($\Delta E = \hbar\Delta\omega$ for photons in a gravity field). The spherically symmetric empty space (Schwarzchild) solution of his equation requires no assumptions about the central point source term $T_{\mu\nu}$, except that in general $T_{\mu\nu} = T_{\nu\mu}$ is essential for internal consistency of Einstein's equation.

Because of its simplicity and elegance, one can easily imagine that Einstein's equation is adequate for all classical gravitational phenomena. The source term $T_{\mu\nu}$ has an "obvious" form only for the special case of dilute dust. Any conjectures about its form in other cases are presently untested and, therefore, unreliable.

Here we shall show that an extension of our previous speculation as to a "natural" gravitation law (Edmonds, 1973) actually becomes more general than Einstein's equation, since it is well defined for $T_{\mu\nu} \neq T_{\nu\mu}$ and reduces to a common consequence of Einstein's equation when $T_{\mu\nu} = T_{\nu\mu}$. There is no compelling reason to believe that $T_{\mu\nu}$ must be symmetric, especially in the quantum world where Einstein's equation has run into considerable difficulty.

Simplicity and elegance are hallmarks of the basic laws of free fermions and bosons in flat space quantum theory. This gives us hope that the same is true of quantum gravity. Our gravity law is structurally fairly simple when compared to the curved space Dirac and Maxwell equations. This is a fair comparison since the universe as a whole is probably curved and unstable in time.

The key to our approach is the development of gravity theory in the same natural hypercomplex number formalism which fits the relativistic quantum theory of electrons and photons. This number system is the 32 element system isomorphic to the algebra of 4 x 4 complex matrices, which in turn is isomorphic the Dirac–Clifford algebra with complex coefficients. This algebra accommodates in a natural way either 3 + 1 or 4 + 1 space-time and our development will apply to both! We have tried to show recently that a subnuclear fourth spatial dimension has a rather natural relationship to rest mass in quantum theory (Edmonds, 1974). Thus quantum gravity at the subnuclear level may well be a 5-space theory. We restrict our present considerations to unquantized equations however.

Our notation has been thoroughly developed in the 1974 papers and the reader is referred there for details. We really need only the facts that $(AB)^\dagger = B^\dagger A^\dagger$ and $(AB)^\wedge = B^\wedge A^\wedge$ for the hypercomplex numbers and that $A = A^\dagger$ implies $A = A^\mu b_\mu$ with $\mu = 0, 1, 2, 3$, for 4-vectors; and $\mu = 0, 1, 2, 3, 4$ for 5-vectors. Here $b_\mu = b_\mu{}^{(0)}(e_0) + b_\mu{}^{(1)}(e_1) + b_\mu{}^{(2)}(e_2) + b_\mu{}^{(3)}(e_3)$ for 4-vectors and has the additional term $b_\mu{}^{(4)}\,(if_0)$ for 5-vectors. Actually there is a sixth term $b_\mu{}^{(5)}(f_0)$ in $A = A^\dagger$ but this is only mixed with the other five under conformal transformations ($SO(4, 2)$). We assume instead that the Lorentz group, $L_e L_e{}^\ddagger = 1(e_0)$, is generalized only to $LL^\wedge = 1(e_0)$, e.g., $SO(4,1)$, as the natural symmetry group in this hypercomplex number approach.

2. *Einstein's Equation Reformulated*

We first rewrite Einstein's gravity law in hypercomplex number form. To do so, we begin with the "definition" of curvature

$$D_{\mu\nu} b^\lambda_{(\alpha)} \equiv (D_\mu D_\nu - D_\nu D_\mu) b^\lambda_{(\alpha)} = -D_{\nu\mu} b^\lambda_{(\alpha)} = R^\lambda_{\ \sigma\mu\nu} b^\sigma_{(\alpha)} \tag{2.1}$$

By multiplying through with $b_\rho^{(\alpha)}$ and summing on (α) we obtain

$$b_\rho^{(\alpha)}(D_{\mu\nu} b^\lambda_{(\alpha)}) = R^\lambda_{\ \sigma\mu\nu} b^\sigma_{(\alpha)} b_\rho^{(\alpha)} = R^\lambda_{\ \sigma\mu\nu}\delta^\sigma_\rho = R^\lambda_{\ \rho\mu\nu} \tag{2.2}$$

Now contracting on λ and μ we obtain

$$R^\lambda_{\ \rho\lambda\nu} \equiv R_{\rho\nu} = b_\rho^{(\alpha)}(D_{\lambda\nu} b^\lambda_{(\alpha)}) \tag{2.3}$$

Again contracting on ρ and ν we have

$$R^\nu_\nu = b^{\nu(\alpha)}(D_{\lambda\nu} b^\lambda_{(\alpha)}) \equiv R \tag{2.4}$$

Straight tensor analysis (no physics) shows that

$$D_\mu(R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R) = 0, R^{\mu\nu} = R^{\nu\mu}, g^{\mu\nu} = g^{\nu\mu} \tag{2.5}$$

Having reached this point, it is then natural to guess that the law of gravity is

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R = \kappa T^{\mu\nu}, \text{then } D_\mu T^{\mu\nu} = 0 \tag{2.6}$$

This is Einstein's equation. Notice that $T^{\mu\nu} = T^{\nu\mu}$ is *necessary* for the source term since the left hand side is symmetric. We can use the $b^\mu_{(\alpha)}$ expressions for $R^{\mu\nu}$ and R along with

$$g_{\mu\nu} \equiv \tfrac{1}{2}[(b_\mu{}^\wedge b_\nu) + (\ )^\wedge] = b^{(\alpha)}_\mu \eta_{(\alpha\beta)} b^{(\beta)}_\nu \equiv (b_\mu | b_\nu) \tag{2.7}$$

to rewrite equation (2.6) entirely in terms of $b^\mu_{(\alpha)}$ and its covariant derivative commutator $D_{\lambda\nu}$. To introduce the hypercomplex numbers $\{e_\alpha\}$ we write

$$b^{(\alpha)}_\mu b_{\nu(\alpha)} = b^{(\alpha)}_\mu \delta^{(\beta)}_{(\alpha)} b_{\nu(\beta)} = b^{(\alpha)}_\mu (e_\alpha | e^\beta) b_{\nu(\beta)} = (b_\mu | b_\nu) \tag{2.8}$$

This puts Einstein's equation in the form

$$(b_\rho | [D_{\lambda\nu} b^\lambda]) - \tfrac{1}{2}(b_\rho | b_\nu)(b^\sigma | [D_{\lambda\sigma} b^\lambda]) = \kappa T_{\rho\nu} \tag{2.9}$$

One of our basic postulates has been that all fundamental laws of nature should be written in terms of hypercomplex numbers with *all* indices summed over. This is proposed as a "substitute" for Einstein's idea that they must be form invariant under arbitrary coordinate transformations. We, therefore, must sum on ρ and ν in equation (2.9). But we have also postulated that natural laws are form invariant under the group symmetry $b_\mu \to b'_\mu \equiv L^\dagger b_\mu L$ with $LL^\wedge \equiv 1(e_0)$, i.e., one replaces all hypercomplex numbers by introducing L's appropriately. The equation must be such that all the L's can be removed from the equation without destroying its equality. (This simple group symmetry is usually looked at from the cumbersome viewpoint of "independent *local* (flat space) Lorentz transformations.") To satisfy these two requirements we multiply $T_{\rho\nu}$ by $b^{\rho\wedge}$ and b^ν and sum on ρ and ν. Then

$$(b^{\rho\prime})^\wedge(T'_{\rho\nu})(b^{\nu\prime}) = (L^\dagger b^\rho L)^\wedge(L^\dagger T_{\rho\nu} L^{\dagger\wedge})(L^\dagger b^\nu L) = L^\wedge(b^{\rho\wedge} T_{\rho\nu} b^\nu)L \tag{2.10}$$

We see that $T_{\rho\nu}$ must transform as $L^\dagger T_{\rho\nu} L^{\dagger\wedge}$ under the L group for from covariance. A special case $T'_{\rho\nu} = T_{\rho\nu} \propto (e_0)$ is consistent with this requirement of L group symmetry. Indeed, equation (2.9), if valid, requires that $T_{\rho\nu}$ be proportional to (e_0) since the left hand side is. (We are assuming that the coupling constant κ is not hypercomplex.)

Equation (2.9) is not a simple equation. One would not likely guess such an equation by looking at the hypercomplex number formulation of space-time. This is contrary to the case of the Dirac and Maxwell equations!

3. *Hypercomplex Number Consequence of Einstein's Equation*

We now multiply equation (2.9) by $b^{\rho\wedge} b^\nu$ and sum on ρ and ν. This gives

$$b^{\rho\wedge} R_{\rho\nu} b^\nu - \tfrac{1}{2}(b_\rho | b_\nu) b^{\rho\wedge} b^\nu R = \kappa b^{\rho\wedge} T_{\rho\nu} b^\nu \tag{3.1}$$

Next we add to this equation its [ ]^ conjugate and obtain

$$R_{\rho\nu}(b^\rho|b^\nu) - \tfrac{1}{2}(b_\rho|b_\nu)(b^\rho|b^\nu)R = \kappa\,\tfrac{1}{2}[b^{\rho\,\wedge}T_{\rho\nu}b^\nu + b^{\nu\,\wedge}T_{\rho\nu}b^\rho] \quad (3.2)$$

We can now use $T_{\rho\nu} = T_{\nu\rho}$ to rewrite the right hand side and obtain

$$R_\nu{}^\nu - \tfrac{1}{2}(b_\nu|b^\nu)R = \kappa b^{\rho\,\wedge}T_{\rho\nu}b^\nu = \kappa(b^\rho|T_{\rho\nu}b^\nu) = \kappa T_\nu{}^\nu \equiv \kappa T \quad (3.3)$$

In 4-space $(b_\nu|b^\nu) = \delta_\nu{}^\nu = 4$ so $R = -\kappa T$. (In 5-space $(b_\nu|b^\nu) = 5$ so $R = -\frac{2}{3}\kappa T$.)

In our 1973 paper, we proposed as the simplest natural gravity law

$$b^{\rho\,\wedge}(D_{\rho\nu}b^\nu) = +\kappa b^{\rho\,\wedge}T_{\rho\nu}b^\nu \quad (3.4)$$

and concluded that it was essentially equivalent to Einstein's equation. Actually, Einstein's equation *contains* it provided $T_{\mu\nu} = T_{\nu\mu}$. The sum over ρ and ν in obtaining equation (3.1) meant that we lost information. Because a sum of several equations is equal, we cannot go backward and assume that the individual parts are separately equal. But the axiom of having laws that are hypercomplex and *invariant* requires no "loose" indices.

4. *Hypercomplex Generalization*

We write equation (3.2) once more in full detail

$$(b^\rho|(b_\rho|[D_{\lambda\nu}b^\lambda])b^\nu) - \tfrac{1}{2}(b_\nu|b^\nu)(b^\rho|[D_{\lambda\rho}b^\lambda]) = \kappa(b^\rho|T_{\lambda\rho}b^\lambda) \quad (4.1)$$

We recall that there are two terms on the left because Einstein believed that $D_\nu T^{\mu\nu} = 0$ should be a physical requirement. A similar approach leads to Maxwell's equation in flat space as we now show. We have $A = A^\mu b_\mu$ and $D = D^\mu b_\mu$ and want a second order differential equation. There are two forms meeting the requirement of L symmetry; $(D|D)^\dagger A$ and $D(D|A)$. Therefore we guess the equation

$$(D|D)^\dagger A + (\text{constant})D(D|A) = \epsilon J \quad (4.2)$$

where ϵ is proportional to charge and $J = J^\mu b_\mu$. By assuming $(D|J) = 0$ we find

$$(D|(D|D)^\dagger A) + (\text{constant})(D|D(D|A)) = \epsilon(D|J) = 0 \quad (4.3)$$

or, using the linearity of the innerproduct and the fact that it is $\propto(e_0)$,

$$(D|D)(D|A) + (\text{constant})(D|D)(D|A) = 0 \quad (4.4)$$

Therefore the (constant) in equation (4.4) is -1 for a conserved current source J. In curved space, however, it is not so simple. We have two kinds of covariant derivative, $D(b_\mu) \neq 0$ and $\mathscr{D}(b_\mu) = D(b_\mu) + \Gamma^\dagger b_\mu + b_\mu\Gamma$ with Γ hypercomplex and *defined* such that $\mathscr{D}(b_\mu) = 0$. Maxwell's equation then, we guess, becomes

$$(\mathscr{D}|\mathscr{D})^\dagger A - \mathscr{D}(\mathscr{D}|A) = \epsilon J \qquad \mathscr{D} = \mathscr{D}^\mu b_\mu = b_\mu\mathscr{D}^\mu \quad (4.5)$$

For the gravity field $\{b_\mu^{(\alpha)}\}$ wave equation we use $D_\mu b_\nu$ rather than $\mathscr{D}_\mu b_\nu$, otherwise we wouldn't have any equation. Actually $(D_\mu D_\nu - D_\nu D_\mu)b^\mu$ seems like a natural form to take since this comes from the curvature idea and partially meets our requirement of summing on all indices. We therefore try to build a second-order differential equation of gravity using $[D_{\lambda\nu}b^\lambda]$ as a "unit." We must sum all indices, satisfy L group symmetry, and make the equation as simple as possible. We therefore would guess (now using equation (4.1) as a hint)

$$c_1 b^{\rho\,\hat{}}[D_{\lambda\nu}b^\lambda]\, b^{\nu\,\hat{}} b_\rho + c_2 b^{\rho\,\hat{}}[D_{\lambda\nu}b^\lambda]\, b_\rho{}^{\hat{}} b^\nu + c_3 b^{\nu\,\hat{}}[D_{\lambda\nu}b^\lambda] = -\kappa b^{\nu\,\hat{}} T_{\lambda\nu} b^\lambda + \mathscr{T} \tag{4.6}$$

Since the left hand side is not equal to its $[\ ]^\hat{}$ we do not need to have $T_{\lambda\rho} = T_{\rho\lambda}$. $\mathscr{T}$ is a hypercomplex source transforming under L according to $\mathscr{T}' = L^\hat{}\mathscr{T}L$. By adding to the equation its $[\ ]^\hat{}$ we obtain

$$\begin{aligned} &c_1 b^{\rho\,\hat{}}\{[D_{\lambda\nu}b^\lambda]b^{\nu\,\hat{}} + b^\nu[\ ]^\hat{}\}b_\rho + c_2 b^{\rho\,\hat{}}[D_{\lambda\nu}b^\lambda]\, b_\rho{}^{\hat{}} b^\nu + \\ &\qquad c_2 b^{\rho\,\hat{}} b_\nu [D_{\lambda\rho}b^\lambda]^\hat{} b^\nu + 2c_3(b^\nu | [D_{\lambda\nu}b^\lambda]) = -2\kappa(b^\nu | T_{\lambda\nu}b^\lambda) + 2(\mathscr{T}|\mathscr{T}) \end{aligned} \tag{4.7}$$

or, on noting that $(A + A^\hat{}) \propto (e_0)$ if $A = b^\lambda b^{\nu\,\hat{}}$, and $b^{\rho\,\hat{}} b_\rho = (b^\rho | b_\rho)$, we get

$$\begin{aligned} &c_1(b^\rho|b_\rho)(b^\nu|[D_{\lambda\nu}b^\lambda]) + \tfrac{1}{2}c_2 b^{\rho\,\hat{}}[[D_{\lambda\nu}b^\lambda]b_\rho{}^\hat{} + b_\nu[D_{\lambda\rho}b^\lambda]]b^\nu \\ &\qquad + c_3(b^\nu|[D_{\lambda\nu}b^\lambda]) = -\kappa(b^\nu|T_{\lambda\nu}b^\lambda) + (\mathscr{T}|\mathscr{T}) \end{aligned} \tag{4.8}$$

We also used

$$(b_\mu | b_\nu) = \tfrac{1}{2}[(b_\mu{}^\hat{} b_\nu) + (\ )^\hat{}] = \tfrac{1}{2}[(b_\mu b_\nu{}^\hat{}) + (\ )^\hat{}] \tag{4.9}$$

Comparing equation (4.8) with R we see that it becomes

$$\begin{aligned} &c_1(b^\rho|b_\rho)R + \tfrac{1}{2}c_2 b^{\rho\,\hat{}}[[D_{\lambda\nu}b^\lambda]b_\rho{}^\hat{} + b_\nu[D_{\lambda\rho}b^\lambda]]b^\nu \\ &\qquad + c_3 R = -\kappa(b^\nu|T_{\lambda\nu}b^\lambda) + (\mathscr{T}|\mathscr{T}) = -\kappa b^{\nu\,\hat{}}\tfrac{1}{2}[T_{\lambda\nu} + T_{\nu\lambda}]b^\lambda + (\mathscr{T}|\mathscr{T}) \end{aligned} \tag{4.10}$$

which is very similar to equation (3.3) only if $c_2 = 0$. We could surmise from Einstein's equation that $c_1 = \frac{1}{2}$ and $c_3 = -1$, but cannot be sure that $c_2 = 0$.

We cannot compare our equation directly with Einstein's equation because his has free indices and ours cannot. At best, we can show that they both lead to some common equation. Our equation (4.10) suggests $R = -\kappa T$ as the classical limit equation. (This contains the Schwarzchild solution in empty space.)

To apply some criterion, analogous to $D_\mu T^{\mu\nu} = 0$, to our gravity equation would require an approach similar to that shown for the Maxwell equation. But what would be reasonable? Should we assume

$$(D|(b^{\nu\,\hat{}} T_{\lambda\nu} b^\lambda)) = 0 \text{ or } (\mathscr{D}|(b^{\nu\,\hat{}} T_{\lambda\nu} b^\lambda)) = 0 \tag{4.11}$$

or something else? We must leave this for the future. At present we only propose that the gravity law is

$$b^{\nu\hat{}}[D_{\lambda\nu}b^{\lambda}] - \tfrac{1}{2}b^{\rho\hat{}}[D_{\lambda\nu}b^{\lambda}]b^{\nu\hat{}}b_{\rho} = \kappa b^{\nu\hat{}}T_{\lambda\nu}b^{\lambda} + \mathscr{T} + (\text{constant})\, b^{\nu\hat{}}\{[D_{\rho\lambda}b^{\rho}]b_{\nu}{}^{\hat{}}\}b^{\lambda} \quad (4.12)$$

Written in this final form with unknown (maybe small—maybe zero) "constant," the "extra" term reminds one of the cosmological constant $\{\Lambda g_{\lambda\nu}\}$ but it is obviously not.

We see that gravity provides a real test of our hypercomplex number axioms about the basic laws. Gravity again proves an important laboratory for examining quantum relativistic laws, even if it is very weak in practice.

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